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Higgs potential in the SU(5) model

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Abstract. We re-analyse the Higgs potential of the SU(5) model where the Higgs sector contains fundamental and adjoint scalar field representations. Unlike previous analyses we discuss the conditions on the coupling constants which ensure that the potential is bounded from below. We examine the stationary points of the potential and derive the conditions under which they are minima. Using these results, we partition the coupling constant space into regions in which each of the different minima is lowest. Finally we state the correct limit in coupling constant space which gives rise to a large gauge hierarchy at the tree level, a result which has been overlooked in previous treatments.

1. Introduction

The unification of the weak and electromagnetic interactions (Weinberg 1967, Salam 1968) within the context of a non-abelian gauge field theory is by now well established. It is also believed that the strong interactions are described by a non-abelian gauge theory, namely QCD (Marciano and Pagels 1978). Following the success of the earlier unification, many attempts have been made in recent years to embed the electroweak gauge group SU(2)×U(1) and the QCD gauge group SU(3) within a single 'unified' gauge group (Pati and Salam 1973, 1974, Georgi and Glashow 1974, Fritsch and Minkowski 1975). Among the models produced, the relative simplicity of the SU(5) model (Georgi and Glashow 1974) has ensured its popularity as a candidate model.

In all unified gauge theories vector meson masses are generated by the Higgs mechanism. This requires the introduction of scalar fields which develop vacuum expectation values (VEVS) as a result of minimising either the tree approximation or the quantum corrected effective potential. For grand unified models, such as the SU(5) model, the symmetry breaking must proceed in at least two stages, for example

$$\mathrm{SU}(5)\to \mathrm{SU}(3)_c\times \mathrm{SU}(2)\times \mathrm{U}(1)\to \mathrm{SU}(3)_c\times \mathrm{U}(1)_{\mathrm{QED}}. \tag{1}$$

This requires the introduction of more than one scalar field representation. For the SU(5) model it has been shown that the desired symmetry breaking can be achieved at the tree level using the fundamental and adjoint representations, i.e. **5** and **24** (Buras *et al* 1978).

The purpose of this paper is to re-examine the tree level Higgs potential of the SU(5) model, for the above choice of Higgs fields. Our interest in this re-examination arises

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for the following reason. Elsewhere we have reported on a proposal of a model as the supersymmetric extension of the SU(5) model (Sherry 1979b). The minimisation of the Higgs potential in the supersymmetric model is not as straightforward as in the conventional SU(5) model (Buras *et al* 1978), as the parameters of the Higgs potential are not all independent, either of each other or of the Yukawa and gauge coupling constants. For this reason one is not free to choose the values of the parameters to give the desired symmetry breaking; rather, we must check if the given values of the parameters allow the desired symmetry breaking to occur. However, this requires that we know beforehand the parameter ranges for which the desired symmetry breaking occurs. To our knowledge this information is not available in the literature. It is our aim in this paper to provide this information.

Our approach to this problem is straightforward. Given the form of the Higgs potential we first discuss the conditions under which it is bounded from below. We next discuss the various calculable stationary points of the potential. The conditions on the parameters are derived by ensuring that each solution exists and is a relative minimum. The latter requires the calculation of the Higgs boson $(\mathrm{mass})^2$ matrix, all of whose non-Goldstone eigenvalues should be positive. The various conditions derived in this manner can be combined to give the partition of the coupling constant space into regions where the different types of symmetry breakdown occur.

An important ingredient in the SU(5) model is the existence of a large gauge hierarchy. This is needed, for example, to suppress proton decay down to the present experimental limits. The supersymmetric SU(5) model must also allow such a large gauge hierarchy to occur. For this reason we investigate the limit which gives rise, at the tree level, to a large gauge hierarchy (Gildener 1976, Buras *et al* 1978). To have a physically meaningful large gauge hierarchy, as recently noted (Sherry 1979a), we must ensure that the limit in which $M_H/M_L \gg 1$ is achieved does not make M_L infinitesimal. Here we denote the heavy vector meson mass by M_H and the light by M_L. The resultant limit, which was overlooked in previous treatments of this question (Gildener 1976, Buras *et al* 1978), clarifies the manner in which a large tree level gauge hierarchy can occur in the SU(5) model. Of course, the stability of this large tree level gauge hierarchy under quantum corrections is still an open question (Gildner 1976, 1979, Namazie and Sayed 1978, Sherry 1979a). However, these ideas must also be applied if computing a large gauge hierarchy with quantum corrections included.

The remainder of this paper is planned as follows. In §2 we introduce the Higgs potential, discuss its boundedness from below and derive the explicit form of the calculable stationary points. In § 3 we calculate the $(\mathrm{mass})^2$ matrix of the Higgs bosons and derive the conditions under which the various stationary points are relative minima. In § 4 we analyse the many conditions derived, give the breakdown of the coupling constant space into regions where each of the different minima is lowest and also examine the large gauge hierarchy limit. Section 5 contains a discussion of the results we have obtained. Finally we include in an Appendix a derivation of some of the conditions necessary to ensure that the Higgs potential is bounded from below.

2. Stationary points of the Higgs potential

The simplest Higgs sector of the SU(5) model consists of two representations, **5** and **24** scalar fields which we denote by H and A respectively. The desired chain of symmetry breaking for this model is that given in (1). It is the purpose of the Higgs potential to yield this symmetry breaking. At the tree level the most general renormalisable Higgs potential is

$$V(H, A) = -\tfrac{1}{2}\mu^2 \mathrm{Tr}A^2 + \tfrac{1}{4}a(\mathrm{Tr}A^2)^2 + \tfrac{1}{2}b\mathrm{Tr}A^4 - \tfrac{1}{2}\nu^2 H^\dagger H + \tfrac{1}{4}\lambda(H^\dagger H)^2 + \alpha H^\dagger H \mathrm{Tr}A^2 + \beta H^\dagger A^2 H \qquad (2)$$

where a reflection symmetry $A \to -A$ is also invoked (Buras *et al* 1978).

Before we proceed to minimise $V(H, A)$, we must restrict the parameters of the potential such that it is bounded from below, as otherwise $V(H, A)$ does not have a minimum value. Ideally one should ask what are the necessary and sufficient conditions on the coupling constants a, b, λ, α and β which guarantee that $V(H, A)$ is bounded from below. In practice, however, this is not feasible. Instead we ask the simpler question: what are the necessary and sufficient conditions such that $V(H, A)$ is bounded from below as any pair of its arguments go to infinity? These conditions are very simply derived, and we refer the interested reader to the Appendix where we briefly sketch the derivation. The analogous conditions, when three or more of the arguments go to infinity, are neither simply derived nor very transparent. The weaker conditions which we use are, as a result, necessary but not sufficient to ensure that $V(H, A)$ is bounded from below.

The conditions give lower bounds on the allowed range of values of the coupling constants which occur in $V(H, A)$. We can express them as follows:

$$\begin{aligned} &a + \tfrac{7}{15}b > 0, \qquad a + \tfrac{7}{3}b > 0, \qquad \lambda > 0, \\ &\min[\alpha, \alpha + \tfrac{1}{2}\beta] > \max[-\tfrac{1}{2}[\lambda(a+b)]^{1/2}, -\tfrac{1}{2}[\lambda(a + \tfrac{3}{2}b)]^{1/2}], \\ &\min[\alpha + \tfrac{3}{10}\beta, \alpha + \tfrac{2}{15}\beta] > -\tfrac{1}{2}[\lambda(a + \tfrac{7}{15}b)]^{1/2}. \end{aligned} \qquad (3)$$

We should emphasise that the necessary and sufficient conditions, while being more stringent, would also be a lot less transparent.

In the absence of the cross terms in $V(H, A)$, i.e. when $\alpha = \beta = 0$, the minimisation of V is straightforward (Li 1974, Buras *et al* 1978). The least symmetric of the possible resulting vacua is invariant under SU(3) × U(1), in which case the symmetry breaking (1) is realised. However, the resulting theory contains a surplus of zero-mass scalars. When we allow α and β to be non-zero, and not too large†, we expect that the vacuum will still be SU(3) invariant. For this reason we look for minima at which H and A take the form

$$H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}}h \end{pmatrix}, \qquad A = v \,\mathrm{diag}\,(1, 1, 1, -\tfrac{3}{2} - \tfrac{1}{2}\epsilon, -\tfrac{3}{2} + \tfrac{1}{2}\epsilon), \qquad (4)$$

with h, v and ϵ real. In the context of the SU(5) model, a large gauge hierarchy at the tree level will require $v \gg h$. In what follows we shall often implicitly assume this

† As we let $\alpha, \beta \to 0$ the minima should go over smoothly to those which occur for $\alpha = \beta = 0$. For α, β 'large' there may exist bifurcating solutions which are singular in α and β. The present analysis is unable to say much about such solutions or how large α and β should be.

property, when both v and h are non-zero. In terms of h, v and ϵ the potential is

$$V(h, v, \epsilon) = -\tfrac{1}{4}(15+\epsilon^2)\mu^2 v^2 + \tfrac{1}{16}a(225+30\epsilon^2+\epsilon^4)v^4 + \tfrac{1}{16}b(105+54\epsilon^2+\epsilon^4)v^4$$
$$-\tfrac{1}{4}\nu^2 h^2 + \tfrac{1}{16}\lambda h^4 + \tfrac{1}{4}\alpha(15+\epsilon^2)h^2 v^2 + \tfrac{1}{8}\beta(3-\epsilon)^2 h^2 v^2. \tag{5}$$

The stationary point equations for $V(h, v, \epsilon)$ are

$$v=0 \quad \text{or} \quad \mu^2 = \tfrac{1}{2}(15a+7b)v^2 + (\alpha+\tfrac{3}{10}\beta)h^2 + (a+\tfrac{9}{5}b)\epsilon^2 v^2$$
$$+\tfrac{1}{30}(a+b)\epsilon^4 v^2 - \tfrac{1}{5}\beta\epsilon h^2 + \tfrac{1}{15}(\alpha+\tfrac{1}{2}\beta)\epsilon^2 h^2 - \tfrac{1}{15}\mu^2\epsilon^2, \tag{6}$$

$$h=0 \quad \text{or} \quad \nu^2 = \tfrac{1}{2}\lambda h^2 + 15(\alpha+\tfrac{3}{10}\beta)v^2 - 3\beta\epsilon v^2 + (\alpha+\tfrac{1}{2}\beta)\epsilon^2 v^2, \tag{7}$$

$$v=0 \quad \text{or} \quad \epsilon=h=0 \quad \text{or} \quad \mu^2 = \tfrac{1}{2}(15a+27b)v^2 + (\alpha+\tfrac{1}{2}\beta)h^2$$
$$-\tfrac{3}{2}\beta h^2/\epsilon + \tfrac{1}{2}(a+b)\epsilon^2 v^2. \tag{8}$$

These equations yield at most seven different types of solution which we can classify as follows:

I	$v \neq 0$,	$\epsilon \neq 0$,	$h \neq 0$,
II	$v \neq 0$,	$\epsilon = h = 0$,	
III	$v \neq 0$,	$\epsilon \neq 0$,	$h = 0$,
IV	$v = 0$,	$\epsilon \neq 0$,	$h = 0$,
V	$v = 0$,	$\epsilon = 0$,	$h \neq 0$,
VI	$v = 0$,	$\epsilon \neq 0$,	$h \neq 0$,
VII	$v = \epsilon = h = 0$.		

For types II to VI these equations can be solved exactly to yield the following solutions.

$$\text{(II)} \qquad v^2 = \frac{2\mu^2}{15a+7b}, \qquad \epsilon = h = 0. \tag{9}$$

For this solution the residual symmetry is SU(3) × SU(2) × U(1).

$$\text{(III)} \qquad v^2 = \frac{\tfrac{1}{2}\mu^2}{10a+13b}, \qquad \epsilon = \pm 5,\ h = 0, \tag{10}$$

$$\text{(V)} \qquad v = \epsilon = 0, \qquad h^2 = 2\nu^2/\lambda. \tag{11}$$

For this solution the residual symmetry is SU(4), while for types IV and VI the equations do not yield a unique solution for ϵ.

In the case of type I stationary points where v, ϵ and h are all non-zero the equations are equivalent to two coupled cubic algebraic equations. It may, however, not be necessary to solve them exactly if α and β are small enough†. In this case we can look for perturbative solutions of the stationary point equations, where the measure of the perturbation, ϵ, is extremely small. If, in fact, we go to the limit $\epsilon^2 v^2 \ll 1$, $\epsilon^4 v^2 \ll 1$, $\epsilon \ll 1$, $\epsilon h^2 \ll 1$, $\epsilon^2 h^2 \ll 1$ but ϵv^2 finite, the stationary point equations simplify to

$$\mu^2 = \tfrac{1}{2}(15a+7b)v^2 + (\alpha+\tfrac{3}{10}\beta)h^2,$$
$$\nu^2 = \tfrac{1}{2}\lambda h^2 + 15(\alpha+\tfrac{3}{10}\beta)v^2 - 3\beta\epsilon v^2, \tag{12}$$
$$\tfrac{3}{2}\beta h^2 = \epsilon(10bv^2 + \tfrac{1}{5}\beta h^2).$$

† See previous footnote.

These equations yield as solutions the following expressions:

$$\epsilon = \frac{3}{20}\frac{\beta}{b}\frac{h^2}{v^2} + O\left(\frac{h^4}{v^4}\right),$$

$$v^2 = \mathscr{P}/\mathscr{N}, \qquad h^2 = \mathscr{S}/\mathscr{N}, \tag{13}$$

where

$$\mathscr{P} = \tfrac{1}{2}(\lambda - 9\beta^2/10b)\mu^2 - (\alpha + \tfrac{3}{10}\beta)\nu^2,$$

$$\mathscr{N} = \tfrac{1}{4}(\lambda - 9\beta^2/10b)(15a + 7b) - 15(\alpha + \tfrac{3}{10}\beta)^2, \tag{14}$$

and

$$\mathscr{S} = \tfrac{1}{2}(15a + 7b)\nu^2 - 15(\alpha + \tfrac{3}{10}\beta)\mu^2.$$

For this solution the residual symmetry is $SU(3) \times U(1)$, and the symmetry breakdown occurs as shown in (1).

Of the four non-trivial stationary points, the type III is not a candidate for the absolute minimum of V. This can be seen quite simply by evaluating $V(h, v, \epsilon)$ at this stationary point. We find that $V(\mathrm{III}) = 5\,\mu^4/(10a + 13b)$ and this is positive provided the solution exists. But, at the relative maximum $V(0, 0, 0) = 0$, so it is clear that, at most, (III) can be a relative, but not absolute, minimum. On the other hand, at the remaining stationary points, II, V and I, the potential is negative.

Before proceeding to the evaluation of the Higgs boson mass matrix, to distinguish between minima, maxima and saddle points, we first consider the range of values of the coupling constants for which these solutions exist. For II and V, the conditions (3) derived earlier ensure their existence. For I it is not so simple. Two cases occur depending on the sign of $(\lambda - 9\beta^2/10b)$. When this is positive, the existence of the type I stationary points is assured when

$$\lambda - 9\beta^2/10b > 0,$$

$$-\frac{1}{2}\left[\left(\lambda - \frac{9}{10}\frac{\beta^2}{b}\right)\left(a + \frac{7}{15}b\right)\right]^{1/2} < \alpha + \frac{3}{10}\beta < \min\left[\frac{\mu^2}{2\nu^2}\left(\lambda - \frac{9}{10}\frac{\beta^2}{b}\right), \frac{\nu^2}{2\mu^2}\left(a + \frac{7}{15}b\right)\right]$$

or

$$\alpha + \frac{3}{10}\beta > \max\left[\frac{\mu^2}{2\nu^2}\left(\lambda - \frac{9}{10}\frac{\beta^2}{b}\right), \frac{\nu^2}{2\mu^2}\left(a + \frac{7}{15}b\right)\right]. \tag{15}$$

On the other hand, when $\lambda - 9\beta^2/10b$ is negative the solution is not a minimum, as we shall see in the next section.

We now know the conditions necessary to ensure the existence of the stationary points, namely (3) and (15). Se we next examine the Higgs boson $(\text{mass})^2$ matrix to discover for which values of the parameters the different stationary points are minima of the potential.

3. Higgs boson masses

The stationary points of the Higgs potential which we found in § 2 may be maxima, minima or saddle points. To distinguish these possibilities we must diagonalise the Higgs boson $(\text{mass})^2$ matrix evaluated at each stationary point. The criterion for minimality is that all the non-Goldstone masses be real, i.e. the non-Goldstone eigenvalues of the (mass)^2 matrix are positive.

To derive the Higgs field mass terms from the potential (2) we use the following field parametrisation:

$$H-\langle H\rangle=\begin{pmatrix} H_i \\ H_4 \\ \frac{1}{\sqrt{2}}\rho\,\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix},$$

$$A-\langle A\rangle=\begin{pmatrix} A(8)_{ij}+\frac{2}{\sqrt{30}}A_0\delta_{ij} & A_{x_i} & A_{y_i} \\ A^{\dagger}_{x_i} & \frac{1}{\sqrt{2}}A_z-\frac{3}{\sqrt{30}}A_0 & A_w \\ A^{\dagger}_{y_i} & A^{\dagger}_w & -\frac{1}{\sqrt{2}}A_z-\frac{3}{\sqrt{30}}A_0 \end{pmatrix}. \tag{16}$$

Here we use latin letters $i, j, \ldots = 1, 2, 3$ to denote SU(3) indices. This parametrisation yields ten SU(3) multiplets. In (16) the VEVs $\langle H\rangle$ and $\langle A\rangle$ are given by equations (4).

It is now straightforward to extract from the Higgs potential the mass terms for the ten relevant SU(3) scalar multiplets. They are

$$\begin{aligned}
&\tfrac{1}{2}[-\mu^2+(\tfrac{15}{2}a+6b)v^2+\alpha h^2]\mathrm{Tr}A(8)^2\\
&+[-\mu^2+\tfrac{1}{2}(15a+7b)v^2+\alpha h^2+2b\epsilon v^2]\boldsymbol{A}^{\dagger}_x\cdot\boldsymbol{A}_x\\
&+[-\mu^2+\tfrac{1}{2}(15a+7b)v^2+(\alpha+\tfrac{1}{2}\beta)h^2-2b\epsilon v^2]\boldsymbol{A}^{\dagger}_y\cdot\boldsymbol{A}_y\\
&+[-\tfrac{1}{2}\nu^2+\tfrac{1}{4}\lambda h^2+(\tfrac{15}{2}\alpha+\beta)v^2]\boldsymbol{H}^{\dagger}\cdot\boldsymbol{H}-(\beta hv/2\sqrt{2})\,(\boldsymbol{H}^{\dagger}\cdot\boldsymbol{A}_y+\boldsymbol{A}^{\dagger}_y\cdot\boldsymbol{H})\\
&+[-\mu^2+\tfrac{1}{2}(15a+27b)v^2+(\alpha+\tfrac{1}{2}\beta)h^2]A^{\dagger}_wA_w\\
&-(3/\sqrt{2})\beta hv(H^{\dagger}_4A_w+A^{\dagger}_wH_4)\\
&+[-\tfrac{1}{2}\nu^2+\tfrac{1}{4}\lambda h^2+\tfrac{15}{2}(\alpha+\tfrac{3}{10}\beta)v^2+\tfrac{3}{2}\beta\epsilon v^2]H^{\dagger}_4H_4\\
&+\tfrac{1}{2}[-\mu^2+\tfrac{3}{2}(15a+7b)v^2+(\alpha+\tfrac{3}{10}\beta)h^2]A_0^2\\
&+\tfrac{1}{2}[-\mu^2+\tfrac{1}{2}(15a+27b)v^2+(\alpha+\tfrac{1}{2}\beta)h^2]A_z^2\\
&+\tfrac{1}{2}[-\tfrac{1}{2}\nu^2+\tfrac{3}{4}\lambda h^2+\tfrac{15}{2}(\alpha+\tfrac{3}{10}\beta)v^2-\tfrac{3}{2}\beta\epsilon v^2]\rho^2+(3/\sqrt{2})\beta hv\rho A_z\\
&+\sqrt{15}[\tfrac{1}{10}\beta h^2-(a+\tfrac{9}{5}b)\epsilon v^2]A_0A_z+\sqrt{30}(\alpha+\tfrac{3}{10}\beta)hv\rho A_0.
\end{aligned}\tag{17}$$

In deriving this expression we have made use of the limit $\epsilon\ll 1$, which is incorporated in the type I stationary point exhibited in § 2, to neglect contributions of the form ϵ^2v^2 and ϵv. For none of the three stationary points, I, II and V, would such terms make a contribution.

Substituting in (17) the explicit solutions derived in § 2 we are led to the following Higgs masses for each type of solution.

Type II:

$$m[A(8)]^2=\tfrac{5}{2}bv^2,\qquad m(A_W)^2=m(A_Z)^2=10bv^2,$$

$$m(A_0)^2=(15a+7b)v^2,\qquad m(\boldsymbol{A}_x)^2=m(\boldsymbol{A}_y)^2=0,$$

$$m(H_4)^2 = m(\rho)^2 = \tfrac{1}{2}[15(\alpha + \tfrac{3}{10}\beta)v^2 - \nu^2],$$
$$m(\boldsymbol{H})^2 = \tfrac{1}{2}[15(\alpha + \tfrac{2}{15}\beta)v^2 - \nu^2]. \tag{18}$$

The mass relations between the various SU(3) multiplets reflect the residual SU(3) × SU(2) symmetry of the vacuum.

Type V:

$$m[A(8)]^2 = m(\boldsymbol{A}_x)^2 = \alpha h^2 - \mu^2,$$
$$m(\boldsymbol{A}_y)^2 = m(A_w)^2 = (\alpha + \tfrac{1}{2}\beta)h^2 - \mu^2,$$
$$m(\boldsymbol{H})^2 = m(H_4)^2 = 0, \qquad m(\rho)^2 = \tfrac{1}{2}\lambda h^2,$$

and in the (A_0, A_z) system the two eigenvalues are

$$\alpha h^2 - \mu^2 \qquad \text{and} \qquad (\alpha + \tfrac{4}{5}\beta)h^2 - \mu^2. \tag{19}$$

Here also the mass relations reflect the residual SU(4) symmetry.

Type I:

$$m[A(8)]^2 = \tfrac{5}{2}bv^2 - \tfrac{3}{10}\beta h^2, \qquad m(\boldsymbol{A}_x)^2 = 0.$$

In the $(\boldsymbol{A}_y, \boldsymbol{H})$ system the two eigenvalues are:

$$0 \qquad \text{and} \qquad -\tfrac{5}{4}\beta v^2 + \tfrac{1}{10}(9\beta/4b - 1)\beta h^2.$$

In the (A_w, H_4) system the two eigenvalues are

$$0 \qquad \text{and} \qquad 10bv^2 + \tfrac{1}{5}(1 + 9\beta/4b)\beta h^2.$$

In the (A_0, A_z, ρ) system the three eigenvalues are, provided $b \neq 0$ and $b \neq 5a$†,

$$\left[\frac{1}{2}\left(\lambda - \frac{9}{10}\frac{\beta^2}{b}\right) - \frac{30(\alpha + 3\beta/10)^2}{15a + 7b}\right]h^2,$$
$$10bv^2 + \tfrac{1}{5}(1 + 9\beta/4b)\beta h^2,$$
$$(15a + 7b)v^2 + \frac{30(\alpha + 3\beta/10)^2}{15a + 7b}h^2. \tag{20}$$

In this case the mass relations reflect the residual SU(3) symmetry.

These results allow us to determine for what values of the coupling constants the three stationary points are relative minima. The conditions are derived by requiring the non-zero Higgs boson $(\text{mass})^2$ values to be positive. For the three solutions we find, over and above the conditions derived in § 2,

II: $$b > 0, \qquad \min[(15\alpha + 2\beta)v^2, (15\alpha + \tfrac{9}{2}\beta)v^2] > \nu^2, \tag{21}$$

V: $$\min[\alpha h^2, (\alpha + \tfrac{4}{5}\beta)h^2] > \mu^2, \tag{22}$$

I: $$\lambda - 9\beta^2/10b > 0, \qquad 5bv^2 - \tfrac{3}{5}\beta h^2 > 0, \qquad b \neq 0,$$
$$10bv^2 + \tfrac{1}{5}(1 + 9\beta/4b)\beta h^2 > 0, \qquad -\tfrac{5}{2}\beta v^2 + \tfrac{1}{5}(9\beta/4b - 1)\beta h^2 > 0. \tag{23}$$

† If $b = 0$, but $a \neq 0$, the eigenvalues are $15av^2 + (2/a)(\alpha + \frac{3}{10}\beta)^2h^2$ and $\pm(3/\sqrt{2})\beta hv\{1 + (b/v)[\ldots]^{-1}\}$. The overall ±sign implies that at least one $(\text{mass})^2$ is negative and such a stationary point cannot be a minimum. On the other hand, if $b = 5a \neq 0$ the eigenvalues are $10bv^2$(twice) and $h^2[\frac{1}{2}(\lambda - 9\beta^2/10b) - (3/b)(\alpha + \frac{3}{10}\beta)^2]$. The positivity conditions derived in this case will be as in the general case, subject to $b = 5a$.

These conditions must now be analysed together with conditions (3) and (15) if we are to have a clear set of conditions on the coupling constants for each of the three minima. We carry through this analysis in the following section, where we also discuss the limit, for the type I minimum, which yields a large gauge hierarchy.

4. Analysis of the conditions and the large gauge hierarchy limit

In the previous sections we have examined some of the stationary points of the SU(5) model Higgs potential $V(H, A)$, and derived the conditions which must be satisfied if they are to be minima. We also derived a partial set of conditions on the parameters of V to ensure that the potential is bounded from below. We now combine together all of the different conditions to list the regions of coupling constant space in which each of the various minima is lowest. Unfortunately, we cannot assert with full rigour that, in each such region, the lowest minimum is the absolute minimum of V because of the approximation we have used to derive the explicit form of the type I solution.

Our first step is to rewrite the boundedness conditions (3). We can partition the coupling constant space into four subspaces according to the signs of β and b. In these four subspaces we find

$$\begin{aligned}
&\beta>0, b>0; && \lambda>0, && a+\tfrac{7}{15}b>0, && \alpha>-\tfrac{1}{2}[\lambda(a+b)]^{1/2},\\
& && && && \alpha+\tfrac{2}{15}\beta>-\tfrac{1}{2}[\lambda(a+\tfrac{7}{15}b)]^{1/2},\\
&\beta>0, b<0: && \lambda>0, && a+\tfrac{7}{3}b>0, && \alpha>-\tfrac{1}{2}[\lambda(a+\tfrac{3}{2}b)]^{1/2},\\
&\beta<0, b<0: && \lambda>0, && a+\tfrac{7}{3}b>0, && \alpha+\tfrac{1}{2}\beta>-\tfrac{1}{2}[\lambda(a+\tfrac{3}{2}b)]^{1/2},\\
&\beta<0, b>0: && \lambda>0, && a+\tfrac{7}{15}b>0, && \alpha+\tfrac{1}{2}\beta>-\tfrac{1}{2}[\lambda(a+b)]^{1/2},\\
& && && && \alpha+\tfrac{3}{10}\beta>-\tfrac{1}{2}[\lambda(a+\tfrac{7}{15}b)]^{1/2}.
\end{aligned} \tag{24}$$

By combining these inequalities with the positivity conditions of § 2 and the mass conditions of § 3, we can give the conditions such that the three stationary points are minima. Let us first examine types II and V.

Type II:

$$\begin{aligned}
& && \lambda>0, \qquad b>0, \qquad a+\tfrac{7}{15}b>0;\\
&\text{if } \beta<0: && \alpha+\tfrac{3}{10}\beta>(\nu^2/2\mu^2)(a+\tfrac{7}{15}b), && \alpha+\tfrac{1}{2}\beta>-\tfrac{1}{2}[\lambda(a+b)]^{1/2},\\
&\text{if } \beta>0: && \alpha+\tfrac{2}{15}\beta>(\nu^2/2\mu^2)(a+\tfrac{7}{15}b), && \alpha>-\tfrac{1}{2}[\lambda(a+b)]^{1/2}.
\end{aligned} \tag{25}$$

Type V:

$$\begin{aligned}
&\beta>0, b>0: && \lambda>0, a+\tfrac{7}{15}b>0, && \alpha>\mu^2\lambda/2\nu^2,\\
&\beta<0, b>0: && \lambda>0, a+\tfrac{7}{15}b>0, && \alpha+\tfrac{4}{5}\beta>\mu^2\lambda/2\nu^2,\\
&\beta>0, v<0: && \lambda>0, a+\tfrac{7}{3}b>0, && \alpha>\mu^2\lambda/2\nu^2,\\
&\beta<0, b<0: && \lambda>0, a+\tfrac{7}{3}b>0, && \alpha+\tfrac{4}{5}\beta>\mu^2\lambda/2\nu^2.
\end{aligned} \tag{26}$$

The type I case requires a little more analysis. Consider the final inequality of (23). If $\beta > 0$, then we find that

$$0 < b < \tfrac{9}{4}\beta \qquad \text{and} \qquad v^2/h^2 < \tfrac{2}{25}(9\beta/4b - 1). \tag{27}$$

The second inequality of (27) implies that $\epsilon \geqslant \frac{5}{6}$. However, the approximation we have used to derive our explicit form of this solution breaks down if ϵ is so large. Therefore we must have $\beta < 0$. Now consider the second inequality of (23) if $b < 0$. Again, we find $v^2/h^2 < 3\beta/25b$, which gives us $\epsilon > \frac{5}{4}$, and the approximation breaks down. Therefore to have solution I defined we must have $\beta < 0 < b$. The combined conditions can be written as

$$\lambda > 9\beta^2/10b, \qquad a + \tfrac{7}{15}b > 0, \qquad \beta < 0 < b,$$

$$-\frac{1}{2}\left[\left(\lambda - \frac{9}{10}\frac{\beta^2}{b}\right)\left(a + \frac{7}{15}b\right)\right]^{1/2} < \alpha + \frac{3}{10}\beta < \min\left[\frac{\mu^2}{2\nu^2}\left(\lambda - \frac{9}{10}\frac{\beta}{b}\right), \frac{\nu^2}{2\mu^2}\left(a + \frac{7}{15}b\right)\right],$$

$$\alpha + \tfrac{1}{2}\beta > -\tfrac{1}{2}[\lambda(a+b)]^{1/2}, \tag{28}$$

and, of course, we must also ensure $v^2 \gg h^2$ so that the approximation we have used is valid. This limit we call the large gauge hierarchy limit.

Before discussing the large gauge hierarchy limit, we can display the regions of coupling constant space in which each of I, II and V is the lowest of the three minima. In fact, there is very little overlap of regions of definition. Types II and V may be defined in the same region. To discriminate between them we note that at these points the potential takes the following values:

$$V(\mathrm{II}) = -\frac{1}{4}\frac{\mu^4}{a + \frac{7}{15}b}, \qquad V(\mathrm{V}) = -\nu^4/4\lambda. \tag{29}$$

The resulting partition of the coupling constant space is depicted in table 1.

Large gauge hierarchy limit: There now merely remains the question of the large gauge hierarchy limit for the type I minimum. The heavy and light gauge boson masses are given by

$$M_{\mathrm{H}}^2 \sim g^2 v^2, \qquad M_{\mathrm{L}}^2 \sim g^2 h^2, \tag{30}$$

provided the approximation is valid. We see then that the approximation we have used is

$$R = M_{\mathrm{H}}^2/\mathrm{M}_{\mathrm{L}}^2 \gg 1. \tag{31}$$

In some treatments of the gauge hierarchy question (31) is used as the gauge hierarchy condition. Explicitly we have

$$R \sim \frac{v^2}{h^2} = \frac{\frac{1}{2}(\lambda - 9\beta^2/10b) - (\nu^2/\mu^2)(\alpha + \frac{3}{10}\beta)}{(\nu^2/2\mu^2)(15a + 7b) - 15(\alpha + \frac{3}{10}\beta)}. \tag{32}$$

Clearly we can achieve $R \gg 1$, within perturbation theory, by letting $(\alpha + \frac{3}{10}\beta)$ go as close to $(\nu^2/2\mu^2)(a + \frac{7}{15}b)$ as we wish. Referring back to the conditions (28), however, we see that we must first require

$$\frac{\nu^2}{2\mu^2}\left(a + \frac{7}{15}b\right) < \frac{\mu^2}{2\nu^2}\left(\lambda - \frac{9}{10}\frac{\beta^2}{b}\right). \tag{33}$$

We have already incorporated these results in table 1.

Table 1.

β	b	Minimum	Qualifying conditions		
>0	<0	V	$\alpha > \dfrac{\mu^2}{2\nu^2}\lambda$		
	>0	V		$a + \dfrac{7}{15}b > \dfrac{\mu^4}{\nu^4}\lambda$	
		II		$a + \dfrac{7}{15}b < \dfrac{\mu^4}{\nu^4}\lambda$	$\alpha + \dfrac{2}{15}\beta > \dfrac{\nu^2}{2\mu^2}\left(a + \dfrac{7}{15}b\right)$
<0	<0	V	$\alpha + \dfrac{4}{5}\beta > \dfrac{\mu^2}{2\nu^2}\lambda$		
	>0	V		$a + \dfrac{7}{15}b > \dfrac{\mu^4}{\nu^4}\lambda$	
		II		$a + \dfrac{7}{15}b < \dfrac{\mu^4}{\nu^4}\lambda$	$\alpha + \dfrac{3}{10}\beta > \dfrac{\nu^2}{2\mu^2}\left(a + \dfrac{7}{15}b\right)$
		I		$\lambda > \dfrac{9}{10}\dfrac{\beta^2}{b}$	$\alpha + \dfrac{3}{10}\beta \leq \dfrac{\nu^2}{2\mu^2}\left(a + \dfrac{7}{15}b\right)$

As was recently pointed out (Sherry 1979a) the condition (31) is necessary, but not sufficient, to yield a physically meaningful large gauge hierarchy. We must also examine M_L^2 and ensure that it does not become infinitesimally small in the above limit. Because of (30) we consider

$$h^2 = \frac{\frac{1}{2}\nu^2(a + \frac{7}{15}b) - \mu^2(\alpha + \frac{3}{10}\beta)}{\frac{1}{4}(\lambda - 9\beta^2/10b)(a + \frac{7}{15}b) - (\alpha + \frac{3}{10}\beta)^2}. \tag{34}$$

It is now straightforward to show that we must supplement the $R \gg 1$ limit by requiring $a + \frac{7}{15}b \to 0$, $\mu^2 \to \infty$, or both, to ensure that h^2 does not vanish. Thus the physical large gauge hierarchy limit is

$$\alpha + \tfrac{3}{10}\beta \to (\nu^2/2\mu^2)(a + \tfrac{7}{15}b)$$
$$\text{and} \qquad a + \tfrac{7}{15}b \to 0 \text{ or } \mu^2 \to \infty \qquad \text{or both.} \tag{35}$$

5. Discussion of results

The initial aim of this investigation was to examine the Higgs potential of the SU(5) model to find its various absolute minima for differing values of the many free parameters. Unfortunately, the defining equations for the least symmetric stationary point are equivalent to two coupled cubic algebraic equations, which cannot be solved exactly (Salmon 1875). However, we have constructed a perturbative solution for this case. When the cross terms in the Higgs potential vanish there are only three possible minima, namely the three we have examined with $\alpha = \beta = 0$. When the coupling constants of the cross term are 'small' we would expect the perturbative solution referred to above to be the lowest of the type I stationary points. Of course, it may be unnecessary to have this cross term 'small', but until we have understood the other non-perturbative solutions of this type we cannot be more quantitative.

As a result, in this paper we have derived, as fully as possible, the conditions on the parameters of the potential necessary for each of the candidate minima to be the lowest. In this analysis we did not use the most stringent conditions on the coupling constants to ensure that the Higgs potential was bounded from below. But since this meant a slightly weaker set of lower limits, it will not affect our results if we ensure that the parameters are kept sufficiently far above these lower limits. As we saw, this can be done for the three minima examined. The results of this analysis are summarised in table 1 where we have given the regions of coupling constant space where the different minima are lowest.

Let us now consider each of the minima in turn. Type V corresponds to SU(5) breaking down to SU(4). In the context of the SU(5) model this minimum is uninteresting. Also we note that the important two conditions for this solution to be lowest are of the form α or $\alpha+\frac{4}{5}\beta>\mu^2\lambda/2\nu^2$ and $a+\frac{7}{15}b>\mu^4\lambda/\nu^4$. If μ^2 is much greater than ν^2 the coupling constants lie well outside the range of validity of perturbation theory.

The type II solution is more interesting from the physical point of view. It corresponds to SU(5) breaking down to SU(3)×SU(2)×U(1), the first step in the desired chain of symmetry breaking for the SU(5) model. The scalar field masses, shown in (18), are all of the same order of magnitude. This minimum at the tree level may be useful as the starting point in the application of Weinberg's approach to the derivation of a large gauge hierarchy (Weinberg 1979). As seen from the final column of table 1, the region where this minimum is lowest borders that where the type I solution is the lowest. The possibility exists that the tree potential minimum may be type II, but the quantum corrections will shift the minimum over the border to a type I, for which a large gauge hierarchy may exist.

If, however, the objective is to have a large gauge hierarchy at the tree level, then we must restrict the parameters of the potential as shown in table 1, and conditions (28), to ensure that the type I minimum is the lowest. As we have also seen, we must supplement the conditions to ensure that a physically meaningful large gauge hierarchy exists, as shown in (35) at the end of the previous section. The effect of the second limit in (35) is that $(\nu^2/2\mu^2)(a+\frac{7}{15}b)$ becomes very small, though positive. Thus the cross-term coupling constant $(\alpha+\frac{3}{10}\beta)$ must be less than, but infinitesimally close to, this very small positive number.

That the cross-term coupling constant should be infinitesimally small to allow a true large gauge hierarchy at the tree level is an unexpected result. It is interesting because originally in deriving a large gauge hierarchy at the tree level one would set this mixing term to zero. It was then pointed out that to allow for more generality one should not restrict the theory in this way (Gildener 1976). The above result shows that if one arranges a true large gauge hierarchy at the tree level, this restriction is forced upon us.

For this solution, in the large gauge hierarchy limit, the various Higgs field masses are, essentially, $\frac{5}{2}bv^2$, $-\frac{5}{4}\beta v^2$, $10bv^2$, $(15a+7b)v^2$ (each of these with negligible $O(h^2)$ corrections) and one of order h^2. If we supplement the large gauge hierarchy limit by $a+\frac{7}{15}b\to 0$, as allowed by (35), there will be two light Higgs fields, namely two independent combinations of (A_0, A_z and ρ). On the other hand, if instead we use $\mu^2\to\infty$, then only one combination of (A_0, A_z, ρ) is light corresponding to the Higgs field of the standard model of the electroweak interactions. The remaining question concerns whether $-\frac{5}{4}\beta v^2$ is large or small. The large gauge hierarchy limit requires that $\alpha+\frac{3}{10}\beta$ be very small and positive. This does not require α and β to be separately infinitesimal so that this mass is, in fact, very large. It is interesting to compare these results with those previously derived (Buras *et al* 1978), where the question of supplementing the condition $R \gg 1$ to yield a large gauge hierarchy was not considered.

Apart from the obvious interest in these results as clarification of the various tree level minima and the limit required to ensure a large tree level gauge hierarchy, they are of interest within a different context. We have recently proposed a model as the supersymmetric extension of the SU(5) model (Sherry 1979b). However, as in all supersymmetric models, the parameters $a, b, \alpha, \beta, \lambda, \mu$ and ν of the corresponding Higgs potential are not all independent—in fact they are not even independent of the gauge and Yukawa coupling constants. The results we have derived in this paper can be directly applied in the minimisation of the Higgs potential in the supersymmetric model. The details of this application of our results will be treated in a separate publication.

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Appendix

The purpose of this Appendix is to derive a partial set of conditions on the coupling constants a, b, λ, α, and β which are necessary if the potential $V(H, A)$ given by equation (2) is to be bounded from below. As explained briefly in the text, we shall not derive the necessary and sufficient conditions. $V(H, A)$ is a quartic polynomial in, essentially, nine SU(3) fields, namely $A(8)$, $\boldsymbol{A}_x$, $\boldsymbol{A}_y$, A_w, A_0, A_z, $\boldsymbol{H}$, H_4 and H_5, which were defined in § 3. Computationally this is a very difficult problem.

Instead we shall examine a much simpler problem which leads to simple conditions on the coupling constants. The price we pay is that the resulting conditions are not sufficient to guarantee $V(H, A)$ bounded from below. However, as all the conditions derived in such analyses yield lower bounds on the coupling constants, what we are doing is accepting *slightly* less stringent bounds at the lower end of the range of allowed values of the coupling constants. What we actually examine is the behaviour of $V(H, A)$ as all pairs of its arguments are separately allowed to vary all over field space. The generic terms which will be relevant in the analysis of the potential are of the form

$$f_1x^4+f_2y^4+f_2x^2y^2. \tag{A1}$$

For a start we must have $f_1>0$ and $f_2>0$. We can then write (A1) as

$$(\sqrt{f_1}x^2-\sqrt{f_2}y^2)^2+[f_3+2(f_1f_2)^{1/2}]x^2y^2. \tag{A2}$$

Here the first term is positive semidefinite, and can vanish as x^2 and y^2 go to infinity. Since (A2) must be bounded for any values of x and y, we deduce that $f_3>-2(f_1f_2)^{1/2}$.

In this way we see that the three conditions

$$f_1>0, \qquad f_2>0 \qquad \text{and} \qquad f_3>-2(f_1f_2)^{1/2}, \tag{A3}$$

are necessary, and sufficient, to ensure that (A1) is bounded from below. We now apply this analysis to the potential $V(H, A)$.

This will entail expanding $V(H, A)$ in terms of the nine SU(3) fields and keeping the quartic terms which contain two, or less, fields. The result of this expansion can be partitioned into two groups of terms. The first of these is independent of the octet field A(8), while in the second all the terms contain the octet field. These two groups of terms are

$$\begin{aligned}
V_1 = {} & (a+b)[(\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x)^2 + (\boldsymbol{A}_y^\dagger . \boldsymbol{A}_y)^2 + (A_w^\dagger A_w)^2] + \tfrac{1}{4}(a+b)A_z^4 \\
& + \tfrac{1}{4}(a+\tfrac{7}{15}b)A_0^4 + \tfrac{1}{4}\lambda(\boldsymbol{H}^\dagger . \boldsymbol{H} + H_4^\dagger H_4 + H_5^\dagger H_5)^2 \\
& + 2a\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x \boldsymbol{A}_y^\dagger . \boldsymbol{A}_y + 2b\boldsymbol{A}_x^\dagger . \boldsymbol{A}_y \boldsymbol{A}_y^\dagger . \boldsymbol{A}_x \\
& + 2(a+b)A_w^\dagger A_w(\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x + \boldsymbol{A}_y^\dagger . \boldsymbol{A}_y) \\
& + (a+\tfrac{7}{15}b)A_0^2(\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x + \boldsymbol{A}_y^\dagger . \boldsymbol{A}_y) + (a+\tfrac{9}{5}b)A_0^2 A_w^\dagger A_w \\
& + (a+b)A_z^2(\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x + \boldsymbol{A}_y^\dagger . \boldsymbol{A}_y + A_w^\dagger A_w) + \tfrac{1}{2}(a+\tfrac{9}{5}b)A_0^2 A_z^2 \\
& + 2\alpha \boldsymbol{H}^\dagger . \boldsymbol{H}(\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x + \boldsymbol{A}_y^\dagger . \boldsymbol{A}_y) + 2\beta(\boldsymbol{H}^\dagger . \boldsymbol{A}_x \boldsymbol{A}_x^\dagger . \boldsymbol{H} + \boldsymbol{H}^\dagger . \boldsymbol{A}_y \boldsymbol{A}_y^\dagger . \boldsymbol{H}) \\
& + [2\alpha A_w^\dagger A_w + (\alpha + \tfrac{2}{15}\beta)A_0^2 + \alpha A_z^2]\boldsymbol{H}^\dagger . \boldsymbol{H} \\
& + [(2\alpha+\beta)(\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x + A_w^\dagger A_w) + 2\alpha \boldsymbol{A}_y^\dagger . \boldsymbol{A}_y \\
& + (\alpha + \tfrac{3}{10}\beta)A_0^2 + (\alpha + \tfrac{1}{2}\beta)A_z^2]H_4^\dagger H_4 \\
& + [(2\alpha+\beta)(\boldsymbol{A}_y^\dagger . \boldsymbol{A}_y + A_w^\dagger A_w) + 2\alpha \boldsymbol{A}_x^\dagger . \boldsymbol{A}_x \\
& + (\alpha + \tfrac{3}{10}\beta)A_0^2 + (\alpha + \tfrac{1}{2}\beta)A_z^2]H_5^\dagger H_5
\end{aligned} \tag{A4}$$

and

$$\begin{aligned}
V_2 = {} & \tfrac{1}{4}a(\mathrm{Tr}A(8)^2)^2 + \tfrac{1}{2}b(\mathrm{Tr}A(8)^4) + (4b/\sqrt{30})A_0(\mathrm{Tr}A(8)^3) \\
& + 2b[\tfrac{1}{5}A_0^2\mathrm{Tr}A(8)^2 + A_x^\dagger A(8)^2 A_x + A_y^\dagger A(8)^2 A_y] \\
& + \tfrac{1}{2}a[A_0^2 + A_z^2 + 2(\boldsymbol{A}_x^\dagger . \boldsymbol{A}_x + \boldsymbol{A}_y^\dagger . \boldsymbol{A}_y + A_w^\dagger A_w)]\mathrm{Tr}\, A(8)^2 \\
& + \alpha(\boldsymbol{H}^\dagger . \boldsymbol{H} + H_4^\dagger H_4 + H_5^\dagger H_5)\mathrm{Tr}\, A(8)^2 + \beta \boldsymbol{H}^\dagger A(8)^2 \boldsymbol{H}.
\end{aligned} \tag{A5}$$

Applying the inequalities (A3) to V_1 will lead to the following conditions:

$$\begin{aligned}
& a+b>0, \qquad a+\tfrac{7}{15}b>0, \qquad \lambda>0, \\
& a+\tfrac{9}{5}b > -[(a+b)(a+\tfrac{7}{15}b)]^{1/2} \\
& \min[\alpha, \alpha+\tfrac{1}{2}\beta] > -\tfrac{1}{2}[\lambda(a+b)]^{1/2}, \\
& \min[\alpha+\tfrac{2}{15}\beta, \alpha+\tfrac{3}{10}\beta] > -\tfrac{1}{2}[\lambda(a+\tfrac{7}{15}b)]^{1/2}.
\end{aligned} \tag{A6}$$

Before we can derive the remaining conditions by applying the inequaliities (A3) to V_2, we should use a particular field parametrisation for the octet $A(8)$. The result is the following pair of conditions, which must be considered together with (A6):

$$a+\tfrac{7}{3}b>0 \qquad \text{and} \qquad \min[\alpha, \alpha+\tfrac{1}{2}\beta] > -\tfrac{1}{2}[\lambda(a+\tfrac{3}{2}b)]^{1/2}. \tag{A7}$$

It is clear that (A6) and (A7) are equivalent to the conditions (3) in § 2, and this is what we wished to achieve.

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